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# Self-adjoint Fuzzy Operator in Fuzzy Hilbert Space and its Properties

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**Abstract**

In this work, we focus our study on adjoint Fuzzy linear operator and self-adjoint Fuzzy linear operator acting on a Fuzzy Hilbert space (FH-Space). We have given several definitions, theorems and discuss in details, The properties of the adjoint and self-adjoint Fuzzy operators in a FH-adjoint Fuzzy operators in a FH-space.

## 1 Introduction

The connotation of a fuzzy norm on a linear space was introduced by Katsaras[1]. Then after many other mathematicians like Felbin [3], Cheng and Mordeson[10] etc, have been taken a definition of fuzzy normed spaces.The concept of fuzzy inner product space (FIP-space) can be deemed as the generalization of the concept of inner product space. The definition of these space was headmost start by R.Biswas[8] and after that according the chronological in [4],[5],[9],[2],[7].

Modulate the definition of fuzzy inner product space (FIP-space) has been insert by M. Goudarzi and S.M.vaezpour in [6],[12]. Also in [6] and [11] given the connotation of a fuzzy Hilbert space (FH-space).

The regulation of the paper is as the following:

Section two includes several preliminary results. In section three we introduce the idea of adjoint fuzzy linear operators, self-adjoint fuzzy linear operators, several theorems, and discusses some properties of such fuzzy operators.

## 2 Preliminaries

***Definition (2.1):*** [6]

A fuzzy inner product space (FIP-space) is a triplet $(X, F, *)$, where $X$ is a real vector space, $*$ is a continuous t-norm, $F$ is a fuzzy set on $X^2 \times \mathbb{R}$ satisfying the following conditions for every $x, y, z \in X$ and $s, r, t \in \mathbb{R}$.

FI-1: $F(x, x, 0) = 0$ and $F(x, x, t) > 0,\ for\ each\ \ t > 0$

FI-2: $F(x, x, t) \neq H(t)$ for some $t \in \mathbb{R}$ if and only if $x \neq 0$, where $H(t) = \begin{cases} 1 & if \quad t > 0 \\ 0 & if \quad t \leq 0 \end{cases}$

FI-3: $F(x,y,t) = F(y,x,t)$

FI-4: For any $\alpha \in \mathbb{R}$, $F(\alpha x,y,t) = \begin{cases} F\left(x,y,\frac{t}{\alpha}\right) & \alpha > 0 \\ H(t) & \alpha = 0 \\ 1 - F\left(x,y,\frac{t}{-\alpha}\right) & \alpha < 0 \end{cases}$

FI-5: $F(x,x,t) * F(y,y,s) \leq F(x+y,x+y,t+s)$

FI-6: $sup_{s+r=t}[F(x,z,s) * F(y,z,r)] = F(x+y,z,t)$

FI-7: $F(x,y,.) : \mathbb{R} \to [0,1]$ is continuous on $\mathbb{R} \setminus \{0\}$

FI-8: $\lim_{t\to+\infty} F(x,y,t) = 1$ .

***Definition (2.2):*** [12]

Let$(E,F,*)$ be a probabilistic inner product space.

1- A sequence $\{x_n\} \in E$ is called $\mathcal{T}$ – converges to $x \in E$ , If for any $\epsilon > 0$ and $\lambda > 0$, $\exists\, N \in Z^+$ , $N = N(\epsilon,\lambda)$ Such that $F_{x_n - x, x_n - x}(\epsilon) > 1 - \lambda$ whenever $n > N$ .

2- A linear functional $f(x)$ defined on $E$ is called $\mathcal{T}_F -$ continuous, if $x_n \overset{\mathcal{T}}{\rightarrow} x$ implies $f(x_n) \overset{\mathcal{T}}{\rightarrow} f(u)$ for any $\{x_n\}$, $x \in E$.

***Theorem (2.3):*** [6]

Let $(X,F,*)$ be a FIP – space, where $*$ is a strong t- norm, and for each $x,y \in X, \sup\{t \in \mathbb{R}: F(x,y,t) < 1\} < \infty$. Define $\langle .,. \rangle : X \times X \to \mathbb{R}$ by $\langle x,y \rangle = \sup\{t \in \mathbb{R}: F(x,y,t) < 1\}$. Then $(X, \langle .,. \rangle)$ is a IP – space (inner product space), so that $(X, \|.\|)$ is a N- space (Normed space), where $\|x\| = \langle x,x \rangle^{1/2}$, $\forall x \in X$ .

***Definition (2.4):*** [6]

Let $(X,F,*)$ be a FIP – space with IP $\langle x,y \rangle = \sup\{t \in \mathbb{R}: F(x,y,t) < 1\}$, $\forall\, x,y \in X$. If $X$ is complete in the $\|.\|$ , then $X$ is called Fuzzy Hilbert – space (FH – space).

***Theorem (2.5):*** [6]

Let $(X,F,*)$ be a FH – space with IP: $\langle x,y \rangle = \{t \in \mathbb{R} : F(x,y,t) < 1\}, \forall\, x,y \in X$. For $x_n \in X$ and $x_n \overset{\|\cdot\|}{\rightarrow} x$ then $x_n \overset{\mathcal{T}_F}{\rightarrow} x$ .

***Proof:*** Since $x_n \overset{\|\cdot\|}{\rightarrow} x$ . Then $\lim_{n\to\infty} \|x_n - x\| = 0$

$$\lim_{n\to\infty} \langle x_n - x, x_n - x \rangle = 0$$

$$\lim_{n\to\infty} \sup\{t \in \mathbb{R}^+ : F(x_n - x, x_n - x, t) < 1\} = 0 \qquad \ldots (2.1)$$

Hence for any $\epsilon > 0$ and $0 < \alpha < 1$ , we have

$$\begin{aligned} &\sup\{t \in \mathbb{R}^+ : F(x_n - x, x_n - x, t) < 1\} \\ &\quad = \sup\{t \in [0,\epsilon] : F(x_n - x, x_n - x, t) < 1\} + \sup\{t \in (\epsilon,\infty) : F(x_n - x, x_n - x, t) < 1\} \quad \ldots(2.2) \\ &\quad \geq \epsilon \sup\{t \in (\epsilon,\infty) : F(x_n - x, x_n - x, t) < 1\} \\ &\quad = \epsilon(1 - F(x_n - x, x_n - x, \epsilon)). \end{aligned}$$

From (2.1)and (2.2), there exist $N(\epsilon,\alpha) \in Z^+$ , If $n > N$ then $F(x_n - x, x_n - x, \epsilon) > 1 - \alpha$

So that $x_n \overset{\mathcal{T}_F}{\rightarrow} x$ .

***Theorem (2.6): (Rise theorem)*** [6],[12]

Let $(X,F,*)$ be a FH- space, For any $\mathcal{T}_F -$ continuous functional, $\exists\, uniqe\ y \in X$ such that for all $x \in X$, we have $g(x) = \sup\{t \in \mathbb{R}: F(x,y,t) < 1\}$ .

## 3 Main Results

This section, we introduce definitions the adjoint fuzzy linear operator and slef-adjont fuzzy linear operator in FH-space as well as some elementary properties of adjoint and self- adjoint fuzzy linear operators in FH-space are presented.

***Theorem (3.1):***

Let $(E,G,*)$be a FIP-space, where $*$ is a strong t- norm and $\sup\{x \in \mathbb{R}: G(u,v,x) < 1\} < \infty$ for all $u,v \in E$, then $\sup\{x \in \mathbb{R}: G(u+v,w,x) < 1\} = \sup\{x \in \mathbb{R}: G(u,w,x) < 1\} + \sup\{x \in \mathbb{R}: G(v,w,x) < 1\}$ $\forall\, u,v,w \in E$.

***Proof:*** By FI-6, for any $\alpha > 0$

$$G(u+v,w,\langle u,w\rangle+\langle v,w\rangle+\alpha) \geq G\left(u,w,\langle u,w\rangle+\frac{\alpha}{2}\right) * G\left(v,w,\langle v,w\rangle+\frac{\alpha}{2}\right)$$
$$= 1 * 1$$
$$= 1\ .$$

This means that

$\sup\{x \in \mathbb{R}: G(u+v,w,x) < 1\} \leq \sup\{x \in \mathbb{R}: G(u,w,x) < 1\} + \sup\{x \in \mathbb{R}: G(v,w,x) < 1\} + \alpha$

Since $\alpha$ arbitrary it implies that

$\sup\{x \in \mathbb{R}: G(u+v,w,x) < 1\} \leq \sup\{x \in \mathbb{R}: G(u,w,x) < 1\} + \sup\{x \in \mathbb{R}: G(v,w,x) < 1\}$ …(3.1)

On the other hand if we choose

$$A = 1 - \left(\left(1 - G(u,w,\langle u,w\rangle - \frac{\alpha}{2})\right) * \left(1 - G(v,w,\langle v,w\rangle - \frac{\alpha}{2})\right)\right)$$

By FI-4 and FI-6 we have :

$$A = 1 - G\left(-u,w,\langle u,w\rangle - \frac{\alpha}{2}\right) * G(-v,w,\langle v,w\rangle - \frac{\alpha}{2})$$
$$\geq G(u+v,w,\langle u,w\rangle+\langle v,w\rangle-\alpha)$$

Since $G\left(u,w,\langle u,w\rangle - \frac{\alpha}{2}\right) < 1$ , $G\left(v,w,\langle v,w\rangle - \frac{\alpha}{2}\right) < 1$ and $*$ be a strong t- norm we have

$G(u+v,w,\langle u,w\rangle+\langle v,w\rangle-\alpha) < 1 \quad \forall\, u,v,w \in E$ and this show that

$\sup\{x \in \mathbb{R}: G(u+v,w,x) < 1\} \geq \sup\{x \in \mathbb{R}: G(u,w,x) < 1\} + \sup\{x \in \mathbb{R}: G(v,w,x) < 1\}$ ….(3.2)

From (3.1) and (3.2) we get :

$\sup\{x \in \mathbb{R}: G(u+v,w,x) < 1\} = \sup\{x \in \mathbb{R}: G(u,w,x) < 1\} + \sup\{x \in \mathbb{R}: G(v,w,x) < 1\}$.

***Remark (3.2):*** Let FB(E) the set of all fuzzy linear operators on $E$ .

***Theorem (3.3):*** **(Adjoint Fuzzy operator in FH- space )**

Let $(E,G,*)$ be a FH- space, Let $s \in FB(E)$ be $\mathcal{T}_F -$ continuous linear functional, then $\exists\, uniqe\ S^* \in FB(E)$ such that $\langle Su,v\rangle = \langle u,S^*v\rangle \quad \forall\ u,v \in E$.

***Proof:***

Fix $v \in E$ , define $F_v: E \to \mathbb{R}$ , by $F_v(u) = \langle Su,v\rangle\ \ \forall\ u \in E$ such that

$F_v(u+w) = F_v(u) + F_v(w)$

$F_v(\alpha u) = \alpha F_v(u) \qquad \forall\ u,v,w\ \in E\ \ ,\alpha\ scalar$ in $\mathbb{R}$.

Also $F_v$ is $\mathcal{T}_F -$ continuous. Then by theorem (2.6), $\exists\, uniqe\ w_v \in E$ such that $F_v(u) = \sup\{x \in \mathbb{R}: G(u,v,x) < 1\}$. Define $S^*: E \to \mathbb{R}$ such that $S_v^* = w_v \quad ,v \in E$

So $\langle Su,v\rangle = F_v(u) = \sup\{x \in \mathbb{R}: G(u,S^*v,x) < 1\}$ .

$S^*$ is linear map: Let $v,w \in E$ and $\alpha$ , $\beta$ are scalars, then

$\langle u,S^*(\alpha v+\beta w)\rangle = \langle Su,\alpha v+\beta w\rangle = \sup\{x \in \mathbb{R}: \quad G(Su,\alpha v+\beta w) < 1]$

By theorem (3.1) and FI-4

$$\begin{aligned}\langle u,S^*(\alpha v+\beta w)\rangle &= \sup\{x \in \mathbb{R}:\ G(Su,\alpha v,x) < 1\} + \sup\{x \in \mathbb{R}:\ G(Su,\beta w,x) < 1\}\\ &= \alpha \sup\{x \in \mathbb{R}:\ G(Su,v,x) < 1\} + \beta \sup\{x \in \mathbb{R}:\ G(Su,w,x) < 1\}\\ &= \alpha\langle Su,v\rangle + \beta\langle Su,w\rangle\\ &= \alpha\langle u,S^*v\rangle + \beta\langle u,S^*w\rangle.\end{aligned}$$

Uniqueness of $S^*$: Let $S_1^*$ , $S_2^*$ be two adjoint Fuzzy operators for $S \in FB(E)$,

$\langle Su,v\rangle = \langle u,S_1^*v\rangle$

$\langle Su,v\rangle = \langle u,S_2^*v\rangle \qquad \forall\ u,v\ \in E$

$\langle u,S_1^*v\rangle = \langle u,S_2^*v\rangle$

$\sup\{x \in \mathbb{R}:\ G(u,S_1^*v,x) < 1\} = \sup\{x \in \mathbb{R}:\ G(u,S_2^*v,x) < 1\}$

$\sup\{x \in \mathbb{R}:\ G(u,S_1^*v - S_2^*v,x) < 1\} = 0$

$G(u,S_1^*v - S_2^*v,x) = 0 \qquad iff \qquad S_1^*v - S_2^*v = 0$

$(S_1^* - S_2^*)v = 0 \qquad \forall\ v \in E$

$S_1^* = S_2^*$

$S^*$ is unique .

***Definition (3.4):***

Let $(E, G, *)$ be a FH- space with IP: $\langle u, v\rangle = \sup\{x \in \mathbb{R}: G(u, v, x) < 1\}$ , $\forall\, u, v \in E$ and let $S \in FB(E)$, then $S$ is self –adjoint Fuzzy operator, if $S = S^*$ where $S^*$ is adjoint Fuzzy operator of $S$ .

***Theorem (3.5):***

Let $(E, G, *)$ be a FH- space with IP: $\langle u, v\rangle = \sup\{x \in \mathbb{R} : G(u, v, x) < 1\}$ and let $S \in FB(E)$, then $S$ is self –adjoint Fuzzy operator.

***Proof:***

Since $E$ is a real vector space and $\langle u, v\rangle = \sup\{x \in \mathbb{R} : G(u, v, x) < 1\} < \infty \quad \forall\, u, v \in E$ , then $\langle Su, u\rangle$ is real for all $u \in E$ , this mean that $\langle Su, u\rangle = \overline{\langle Su, u\rangle} = \langle u, Su\rangle$

$\sup\{x \in \mathbb{R}: G(Su, u, x) < 1\} = \sup\{x \in \mathbb{R}: G(u, Su, x) < 1\} = \sup\{x \in \mathbb{R}: G(S^*u, u, x) < 1\}$

$\sup\{x \in \mathbb{R} : G(Su - S^*u, u, x) < 1\} = 0$

$(S - S^*)u = 0 \qquad \forall\, u \in E$

$S = S^* \qquad \Longrightarrow\; S$ is self- adjoint Fuzzy operator .

***Theorem (3.6):***

Let $(E, G, *)$be a FH- space with IP: $\langle u, v\rangle = \sup\{x \in \mathbb{R} : G(u, v, x) < 1\}, \forall u, v \in E$ and let $S^*$ be the adjoint Fuzzy operator of $s \in FB(E)$, then :

i. $(S^*)^* = S$

ii. $(\alpha S)^* = \alpha S^*$

iii. $(\alpha S + \beta T)^* = \alpha S^* + \beta T^*$ where $\alpha, \beta$ are scalars and $T \in FB(E)$.

iv. $(ST)^* = T^*S^*$.

***Proof:***

i) By FI-3 and theorem (3.2)

$\langle Su, v\rangle = \langle u, S^*v\rangle = \sup\{x \in \mathbb{R} : G(u, S^*v, x) < 1\} = \sup\{x \in \mathbb{R} : G(S^*v, u, x) < 1\}$

$= \langle S^*v, u\rangle = \langle v, (S^*)^*u\rangle = \sup\{x \in \mathbb{R} : G(v, (S^*)^*u, x) < 1\} = \sup\{x \in \mathbb{R} : G((S^*)^*u, v, x) < 1\}$

$= \langle (S^*)^*u, v\rangle = \langle (S - (S^*)^*)u, v\rangle = 0$

$(S - (S^*)^*)u = 0 \quad \forall u \in E \;\Rightarrow\; S = (S^*)^*$ .

ii) By theorem (3.1),(3.2) , FI-3, FI-4 and linearity of operators, we get:

$\langle \alpha Su, v\rangle = \sup\{x \in \mathbb{R} : G(u, (\alpha S)^*v, x) < 1\} = \langle u, (\alpha S)^*v\rangle$ ...(3.3)

And

$\langle \alpha Su, v\rangle = \sup\{x \in \mathbb{R} : G(\alpha Su, v, x) < 1\}$

$= \alpha \sup\{x \in \mathbb{R} : G(Su, v, x) < 1\}$

$= \alpha\langle Su, v\rangle = \alpha \sup\{x \in \mathbb{R} : G(u, S^*v, x) < 1\}$

$= \alpha\langle u, S^*v\rangle$ …(3.4)

From (3.3) and (3.4) we get:

$\langle u, (\alpha S)^*v\rangle - \alpha\langle u, S^*v\rangle = 0$

$\langle u, ((\alpha S)^* - \alpha S^*)v\rangle = 0 \qquad \forall\; u, v \in E$

$((\alpha S)^* - \alpha S^*)v = 0 \qquad \forall\; v \in E$

$(\alpha S)^* = \alpha S^*$ . …(3.5)

iii) By theorem (3.1),(3.2) , FI-3 and FI-4 we get:

$\langle (\alpha S + \beta T)u, v\rangle = \sup\{x \in \mathbb{R} : G(u, (\alpha S + \beta T)^*v, x) < 1\}$

$= \langle u, (\alpha S + \beta T)^*v\rangle$ …(3.6)

And

$\langle (\alpha S + \beta T)u, v\rangle = \sup\{x \in \mathbb{R} : G((\alpha S + \beta T)u, v, x) < 1\}$

$= \sup\{x \in \mathbb{R} : G(\alpha Su, v, x) < 1\} + \sup\{x \in \mathbb{R} : G(\beta Tu, v, x) < 1\}$

$= \langle \alpha Su, v\rangle + \langle \beta Su, v\rangle$

$= \sup\{x \in \mathbb{R} : G(u, (\alpha S)^*v, x) < 1\} + \sup\{x \in \mathbb{R} : G(u, (\beta T)^*v, x) < 1\}$

$= \sup\{x \in \mathbb{R} : G(u, ((\alpha S)^* + (\beta T)^*)v, x) < 1\}$

$= \langle u, ((\alpha S)^* + (\beta T)^*)v \rangle$ …(3.7)

From (3.6) and (3.7) we get:

$\langle u, (\alpha S + \beta T)^* v \rangle - \langle u, ((\alpha S)^* + (\beta T)^*)v \rangle = 0$

$\langle u, ((\alpha S + \beta T)^* - ((\alpha S)^* + (\beta T)^*))v \rangle = 0 \qquad \forall\ u, v\ \in E$

$((\alpha S + \beta T)^* - ((\alpha S)^* + (\beta T)^*))v = 0 \qquad \forall\ v \in\ E$

$(\alpha S + \beta T)^* = (\alpha S)^* + (\beta T)^*$ …(3.8)

From (3.5) and (3.8) we get : $(\alpha S + \beta T)^* = \alpha S^* + \beta T^*$.

iv) By theorem (3.1),(3.2) and FS3 we get:

$\langle STu, v \rangle = \sup\{x \in \mathbb{R} : G(u, (ST)^* v, x) < 1\}$

$= \langle u, (ST)^* v \rangle$ ……(3.9)

$\langle STu, v \rangle = \langle Tu, S^* v \rangle = \sup\{x \in \mathbb{R} : G(Tu, S^* v, x) < 1\} = \sup\{x \in \mathbb{R} : G(u, T^* S^* v, x) < 1\}$

$= \langle u, T^* S^* v \rangle$ …..(3.10)

From (3.9) and (3.10) we get:

$\langle u, (ST)^* v \rangle - \langle u, T^* S^* v \rangle = 0$

$\langle u, ((ST)^* - (T^* S^*))v \rangle = 0 \qquad \forall\, u, v\ \in E$

$((ST)^* - (T^* S^*))v = 0 \qquad \forall\ \ v \in E$

$(ST)^* = (T^* S^*)$. ■

***Theorem(3.7):***

Let $(E, G, *)$be a FH- space with IP:$\langle u, v \rangle = \sup\{x \in \mathbb{R} : G(u, v, x) < 1\}, \forall u, v \in E$ and let $S \in FB(E)$ then, $\|Su\| = \|S^* u\| \qquad for\ all \quad u \in E$

***Proof:***

By theorem (3.3) $S$ is self- adjoint Fuzzy operator then $SS^* = S^* S$

$\langle S^* Su, u \rangle = \langle SS^* u, u \rangle \qquad \forall\ u \in E$

$\sup\{x \in \mathbb{R} : G(S^* Su, u, x) < 1\} = \sup\{x \in \mathbb{R} : G(SS^* u, u, x) < 1\}$

$\sup\{x \in \mathbb{R} : G(Su, Su, x) < 1\} = \sup\{x \in \mathbb{R} : G(S^* u, S^* u, x) < 1\}$

$\langle Su, Su \rangle = \langle S^* u, S^* u \rangle$

$\|Su\| = \|S^* u\|$ .

## Conclusion

In this work:

1- We attempted to insert feasible definitions of the adjoint Fuzzy linear operator and self-adjoint Fuzzy linear operator in FH- space

2- We attempted to prove some properties of the adjoint and self –adjoint Fuzzy operators in FH-space.

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